Liquid Capital and Market Liquidity

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Abstract

Liquidity, as a description of an agent's asset holdings, refers to the ease with which these assets can be converted directly to goods and services. As a description of market conditions, liquidity refers to the willingness of agents to accomodate the trading needs of others. This paper views the former notion as a technological property of real assets and the latter as an endogenous property of financial equilibrium, and describes a channel by which the two are linked. When agents hold more wealth in technologically liquid investments, a marginal adjustment to portfolio holdings alters discount rates less, causing a smaller price impact. Thus, even without intermediaries or frictions, the stock of transformable capital may be a crucial determinant of the resilience of financial markets.

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JEL CLASSIFICATIONS: D91, E21, E44, E52, G12

1 Introduction

Consider an economy in which agents can choose to save physical capital in investments of varying degrees of transformability. Barriers to capital transformation, due to adjustment costs or irreversibility, play a central role in many models. When there are differences across sectors in these barriers, the relative allocation to more or less convertible forms of capital may have broad macroeconomic consequences. This paper asks how endogenous

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variation in the aggregate degree of transformability affects securities prices, returns dynamics, and market liquidity.

Understanding the latter linkage provides the primary motivation, since the aggregate transformability of capital corresponds to a second, economically distinct, notion of liquidity. When an agent – an individual, a firm, or a country – holds a large fraction of its wealth in forms that can be directly converted to current consumption it is said to be liquid. Here I view that liquidity as a real, technological property that applies to the physical capital itself, and not as a property of a secondary market for *claims to* that capital.

A secondary market is said to be liquid when agents can sell *or buy* with little price concession. Here the term does not refer to a property of the security being bought or sold (i.e. to its cashflows) nor to the physical assets to which it is a claim. Rather, it describes the willingness of some agents to accommodate the portfolio rebalancing demands of others.

The question is, are these two types of liquidity linked? Understanding the dynamics of market liquidity and liquidity risk is a central topic in financial research and is of significant concern to investors as well as policy makers. The drivers of systematic fluctuations in liquidity (and, in particular, of systematic declines) are not well understood. A natural hypothesis is that a contributing factor is the quantity of available capital.

I interpret "available capital" not in terms of some particular form of credit but instead in terms of primitive properties of investments. In the context of a simple two-sector representative-agent model, the paper describes a mechanism linking the real liquidity of the economy to the liquidity of its asset market. This linkage is not related to the role of intermediaries or the the institutional arrangements of trade. When agents hold more wealth in technologically liquid form, a marginal purchase of risky claims alters discount rates less, inducing a smaller price impact, meaning more liquid markets. When the stock of transformable wealth is low, a marginal purchase of shares will entail a relatively large sacrifice of current consumption, which will raise discount rates and depress prices. I argue that this effect is large and could account for a significant portion of observed fluctuations in market liquidity.

Formally, the notion of market illiquidity used here corresponds to the steepness of a representative agent's demand curve for shares. This idea is due to Pagano (1989). The precise definition follows Johnson (2006), which treats arbitrary endowment economies. The present paper further extends the applicability of this measure to an economy with an endogenously varying capital stock. The argument shows that the elasticity of asset prices with respect to a marginal perturbation in share holdings is higher when such a trade induces greater intertemporal substitution in consumption. That, in turn, occurs more readily when liquid asset holdings are low because the marginal propensity to consume is higher.

The theory developed in the paper stands in contrast to an alternative view of the relation between liquid capital and market liquidity. It is widely believed that the "supply of liquidity" – meaning cash – influences the resilience of secondary markets through the financial constraints of specialist providers of two-way prices. In this view, market makers possess some production function for the creation of market liquidity, and also face imperfect financial markets. When central banks "supply liquidity" in times of stress to promote the orderly functioning of financial markets, they often cite this view of liquidity determination.

The analysis presented in this paper suggests that the presence of contracting frictions in the supply of credit to intermediaries may not be the whole story. The model, in fact, implies that there should be a strong correlation between measures of funding liquidity and measures of market resilience. But this may not tell us anything about the existence (or severity) of such frictions. More broadly, limited willingness to trade when capital is limited does not imply systemic failure of credit markets. It may be an equilibrium phenomenon.

Beyond liquidity, a secondary contribution of the paper is to delineate further consequences of fluctuations in aggregate transformability for asset pricing. Much popular attention, and numerous models, have focused on the likely effects of increases or decreases in the supply of liquid capital on asset markets. The model directly addresses the connection between liquidity – in real terms – and both volatility and expected returns. In particular, I show that, because of its consumption buffering role, a rising stock of transformable wealth also implies that volatility and risk premia should endogenously decline. As with market liquidity, these associations are not due to any irrational distortion or agency problems of intermediaries.

The outline of the paper is as follows. In the next section, I introduce the economic setting. This is perhaps the simplest model one can study in which agents choose investments with different degrees of adjustability. I describe the economy formally and discuss equilibrium properties of consumption and savings. Section 3 computes asset prices, expected returns, and volatility. Numerical examples illustrate the dependence of all of these on the degree of liquid capital. In Section 4, I define the concept of market liquidity and show how to compute it in this model. I analyze the determinants of this quantity and highlight the intuition behind them. The primary result is that market liquidity increases with the level of liquid asset holdings. Section 5 addresses some issues of interpretation. I suggest proxies for the degree of real liquidity and discuss their relation to other notions of the supply of liquid capital. I also consider the impact of interventions designed to raise market liquidity. A final section summarizes the paper's contribution and concludes.

2. An Economy with Time-Varying Liquid Capital

This section develops a standard model in which agents choose how much of their wealth to hold in transformable, or liquid, form. Liquid capital is characterized by a technological property: it is freely and immediately convertible into consumption. Although there is no fiat money in the model, wealth held in this form is cash-like in the sense that it can be exchanged *directly* for needed goods and services. By contrast, non-liquid capital is costly (or impossible) to physically adjust.[1] In solving the model, the degree of real (or technological) liquidity becomes the main state variable driving consumption and savings. The goal of this section is to characterize the dynamics of that variable, and hence of consumption.

The setting is as follows. Time is discrete and an infinitely lived representative agent has constant relative risk aversion (CRRA) preferences over consumption of a single good. The agent receives a stochastic stream, D_t, of that good in each period from an endowment asset. In addition, the agent has access to a second investment technology whose capital stock can be altered freely each period, and which returns a gross rate, $\widehat{R}$, which may also be random. The capital stock of the endowment asset can be neither increased nor decreased. Agents can only alter their savings via the liquid investment.

Models with an elastically supplied storage technology are common in asset pricing. This investment opportunity is then usually interpreted as (one-period) government bonds. Here the technology is not financial but real. In the proverbial "fruit tree" interpretation of the Lucas (1978) economy, the liquid capital stock just corresponds to stored fruit. The most literal interpretation of this component of wealth, then, is as commodity inventories. Alternatively, one can simply view the setting as a two-sector stochastic

[1] In a disaggregated economy, *claims* to such capital may be traded, and thus converted to consumable form. This operation is contingent on access to a secondary market, and, in any case, does not alter the aggregate holdings of the non-liquid asset.

growth model, where the sectors differ in the cost of adjusting their physical capital. The present case takes the distinction between these two to the extreme for expositional simplicity. One could interpret its dynamics as describing the short- to medium-term adjustment of an economy in which all physical investment can be altered, but with some involving greater difficulty (or longer time) than others. The model generalizes the pure endowment economy in which *no* sector's capital can be adjusted, and in which fluctuations in savings demand are reflected only in the riskless rate.

The asset market implications of a variable stock of liquid savings in a CRRA economy have not, to my knowledge, been previously analyzed.[2] Such buffer-stock models have been extensively studied in the consumption literature, however. (See Deaton (1991) and Carroll (1992).) The focus in the present work is on the the endowment stream, which I interpret as dividends, but which in that literature is interpreted as labor income. Since labor income is not traded, its price is not a primary object of study. Labor models also sometimes impose the constraint that investment in the savings technology must be positive, ruling out borrowing. This is intuitively sensible as a property of aggregate savings as well, but need not be imposed here, because it will hold endogenously anyway in the cases considered below.[3] Moreover, it is worth pointing out that there are also no financial constraints in the model. Agents in this economy may write any contracts and trade any claims with one another.

Setting the notation, the representative agent's problem is to choose a consumption policy, C_t, to maximize

$$J_t = \mathrm{E_t}\left[\sum_{k=1} \beta^k \, \frac{C_{t+k}^{1-\gamma}}{1-\gamma}\right],$$

where $\beta \equiv e^{-\phi}$ is the subjective discount factor, and I have set the time interval to unity for simplicity. A key variable is the total amount of goods, G_t, that the agent could consume at time t, which is equal to the stock of savings carried into the period plus new dividends received:

$$G_t = \widehat{R}_t(G_{t-1} - C_{t-1}) + D_t \equiv K_t + D_t,$$

where K_t denotes the beginning-of-period supply of available goods. The end-of-period stock of goods invested in the transformable technology will be denoted $B_t \equiv G_t - C_t$.

[2]The literature typically employs riskless storage technology in conjunction with constant absolute risk aversion (CARA) utility. CARA utility is unsuitable here because it necessarily implies that demand for the risky asset is independent of the level of savings or liquid wealth.

[3]With lognormal dividend shocks and CRRA preferences, agents will never borrow in a finite horizon economy. The policies below are limits of finite horizon solutions. When these limits exist, and appropriate transversality conditions are satisfied, they are also solutions to the infinite horizon problem.

It is also useful to define the income each period $I_t \equiv (\widehat{R}_t - 1)B_{t-1} + D_t = (\widehat{R}_t - 1)(G_t - D_t)/\widehat{R}_t + D_t$. Although this quantity plays no direct role, it helps in understanding savings decisions.

To take the simplest stochastic specification, I assume D_t is a geometric random walk:

$$D_{t+1} = D_t \ \widetilde{R}_{t+1}, \quad \log \widetilde{R}_{t+1} \sim \mathcal{N}(\tilde{\mu} - \frac{\tilde{\sigma}^2}{2}, \tilde{\sigma}^2)$$

so that dividend growth is *i.i.d.* with mean $e^{\tilde{\mu}}$.[4] I also assume *i.i.d.* lognormal returns to liquid capital:

$$\log \widehat{R} \sim \mathcal{N}(\hat{\mu} - \frac{\hat{\sigma}^2}{2}, \hat{\sigma}^2).$$

Note that storage of goods could be costly on average if $\hat{\mu} < 0$.

The model admits a useful simplification under the assumption that the return *spread* $\widetilde{Q}_t \equiv \widetilde{R}_t/\widehat{R}_t$ is independent of $\widehat{R}_t$. In that case, it turns out that the model is characterized by just five parameters: β, γ, μ, σ, and R, where R is the certainty equivalent return on the liquid asset,

$$R = e^r \equiv \left(\mathrm{E}\widehat{R}^{1-\gamma}\right)^{\frac{1}{1-\gamma}} = e^{\hat{\mu} - \frac{1}{2}\gamma\hat{\sigma}^2}$$

and $\mu = r + \tilde{\mu} - \hat{\mu}$ and $\sigma^2 = \tilde{\sigma}^2 - \hat{\sigma}^2$, which is assumed to be positive. With these definitions, $\mu - r$ is the expected growth of $\widetilde{Q}$, and σ is its volatility.

The state of the economy is summarized by G_t and D_t which together determine the relative scale of the endowment stream. It is easy to show that the optimal policy must be homogeneous of degree one in either variable. So it is convenient to define

$$v_t = D_t/G_t$$

which takes values in $[0, 1]$. This variable can also be viewed as summarizing the real illiquidity of the economy. As $v_t \to 0$, the endowment stream becomes irrelevant and all the economy's wealth is transformable to consumption whenever desired, and income fluctuations can be easily smoothed. As $v_t \to 1$ on the other hand, all income effectively comes from the endowment asset whose capital stock cannot be adjusted. Since liquid holdings are small, agents have little ability to dampen income shocks.

The agent's decision problem is to choose how much of his available goods to consume at each point in time. The solution can be characterized by the first order condition.

[4] Including a transient component in dividends will not alter the features of the model under consideration here.

Invoking homogeneity, the optimal policy can be written $C(G, v) = G\ h(v)$, and the first order condition requires

$$[h(v_t)/v_t]^{-\gamma} = R\beta \mathrm{E_t}\left[\left[\left(h(v_{t+1})\widetilde{Q}_{t+1}\right)/v_{t+1}\right]^{-\gamma}\right].$$

This expression, which just writes marginal utility growth using the definitions above, may be solved iteratively for h using the law of motion for v:

$$v_{t+1} = v_t\widetilde{Q}_{t+1}/((1 - h(v_t)) + v_t\widetilde{Q}_{t+1}). \tag{1}$$

What does the agent's optimal consumption policy look like? Intuition would suggest that he will optimally consume less of the total, G_t, when v is low than when v is high, since, in the former case consuming the goods amounts to eating the capital base, whereas in the latter case, G_t is mostly made up of the income stream, D_t, which can be consumed with no sacrifice of future dividends. This property is, in fact, true very generally.

Proposition 2.1 *Assume the infinite-horizon problem has a solution policy $h \equiv C/G = h(v)$ such that $h(v) < 1$. Then,*

$$h' > 0.$$

Note: all proofs appear in Appendix A.

This result can also be understood by noting that $\partial C(D, G)/\partial D = h'$ so that the assertion is only that consumption increases with dividend income, which is unsurprising because shocks to D are permanent. Moreover, the result holds for much more general preferences. This follows from the results of Carroll and Kimball (1996) who show that $\partial^2 C/\partial G^2 < 0$ whenever $u'''\, u'/[u'']^2 > 0$. Concavity implies that $0 < C - G\ \partial C/\partial G$ and the latter quantity divided by D also equals h'.

That h rises with v is essentially the only feature of the model that is necessary for the subsequent results. Further useful intuition about the dynamics of the model can be gained by considering how consumption behaves at the extreme ranges of the state variable v_t.

In the limit as $v_t \to 1$ the economy would collapse to a pure endowment one (Lucas, 1978) if the agent consumed all his dividends, i.e. if $h(1) = 1$. This will not happen if the riskless savings rate available exceeds what it would be in that economy. That is, if

$$\frac{1}{R^f} < \mathrm{E_t}\left[\beta\left[\frac{C_{t+1}}{C_t}\right]^{-\gamma}|h(1)=1\right] = \mathrm{E_t}\left[\beta\left[\frac{G_{t+1}h(v_{t+1})}{G_t h(1)}\right]^{-\gamma}|h(1)=1\right] = \mathrm{E_t}\left[\beta\left[\widetilde{R}_{t+1}\right]^{-\gamma}\right] \tag{2}$$

then the agent's marginal valuation of a one-period riskless investment, assuming no savings, exceeds the cost of such an investment. So not saving cannot be an equilibrium, and the conclusion is that the inequality (2) implies $h(1) < 1$. Under lognormality it is straightforward to show that $r^f \equiv \log R^f = r - \frac{1}{2}\gamma\widehat{\sigma}^2$ and then the condition for $h(1) < 1$ is just $r^f > \phi + \gamma\widetilde{\mu} - \gamma(1+\gamma)\widetilde{\sigma}^2/2$, the right-hand expression being the familiar interest rate in the Lucas (1978) economy.[5]

As dividends get small relative to stored goods, $v_t \to 0$, the economy begins to look like a standard one-sector growth model with *i.i.d.* productivity and no adjustment costs. For such an economy, it is straightforward to show that the optimal consumption fraction is $h_0 \equiv (R - (R\beta)^{1/\gamma})/R$. Now if the agent's consumption fraction approached this limit (which it does, $h()$ is continuous at zero[6]), his liquid balances would grow at certainty equivalent rate approaching $(R\beta)^{1/\gamma}$. If this rate exceeds the rate of dividend growth, then v_t will shrink further. However, in the opposite case, the agent is dissaving sufficiently fast to allow dividends to catch up. Hence v_t will tend to rebound. Thus the more interesting case is when

$$(R\beta)^{1/\gamma} < \mathrm{E_t}\left[\widetilde{R}_{t+1}\right] \tag{3}$$

or $r < \phi + \gamma\mu$ under lognormality. A somewhat stronger condition would be that the agent actually dissaves as $v_t \to 0$. This would mean consumption $G_t h_0$ exceeds (certainty equivalent) income $(R-1)(G_t - D_t)/R + D_t$ or, at $v = 0$, $h_0 < (R-1)/R$, which implies $(R\beta)^{1/\gamma} < 1$ or simply $r < \phi$.

With (2) and (3), then, the state variable v_t is mean-reverting.[7] This requirement is not necessary for any of the results on prices or liquidity. However it makes the equilibrium richer. As an illustration, I solve the model for the parameter values shown

[5]The opposite inequality to (2) is sometimes imposed in the buffer stock literature to ensure dissavings as $v_t \to 0$. In that case, the specification of D_t is altered to include a positive probability that $D_t = 0$ each period. This assures that the agent will never put $h = 1$.

[6]This is shown in Carroll (2004) under slightly different conditions. Modification of his argument to the present model is straightforward.

[7]Stationarity of v implies stationarity of consumption as a fraction of available wealth $C/G = h(v)$. Both are general properties under the model of Caballero (1990) who considers CARA preferences. Clarida (1987) provides sufficient conditions under CRRA preferences when dividends are *i.i.d.* Szeidl (2002) generalizes these results to include permanent shocks. No assumption about the long-run properties of the model are used in the subsequent analysis.

in Table 1.

Table 1: *Baseline Parameters.*

PARAMETER	NOTATION	VALUE
coefficient of relative risk aversion	γ	6
subjective discount rate	$\phi = -\log \beta$	0.05
certainty equivalent liquid return	$r = \log R$	0.02
expected excess dividend growth rate	μ	0.04
volatility of excess dividend growth	σ	0.14
time interval	Δt	1 year

The parameters are fairly typical of calibrations of aggregate models when the endowment stream is taken to be aggregate dividends (with R approximating the real interest rate), and the preference parameters are all in the region usually considered plausible. Figure 1 plots the solution for the consumption function for these parameters. Analytical solutions are not available. However, as the proposition above indicated, h is increasing and concave. Also plotted is (certainty equivalent) income as a fraction of G, which is $(R - 1 + v)/R$. This shows the savings behavior described above: for small values of v agents dissave, whereas they accumulate balances whenever v exceeds about 0.12.

Consumption dynamics in this model are driven by the joint influences of exogenous shocks to available goods, G, and by the evolution of the endogenous variable v via the marginal propensity to consume those goods, $h(v)$. Shocks to v come from the exogenous processes as well.

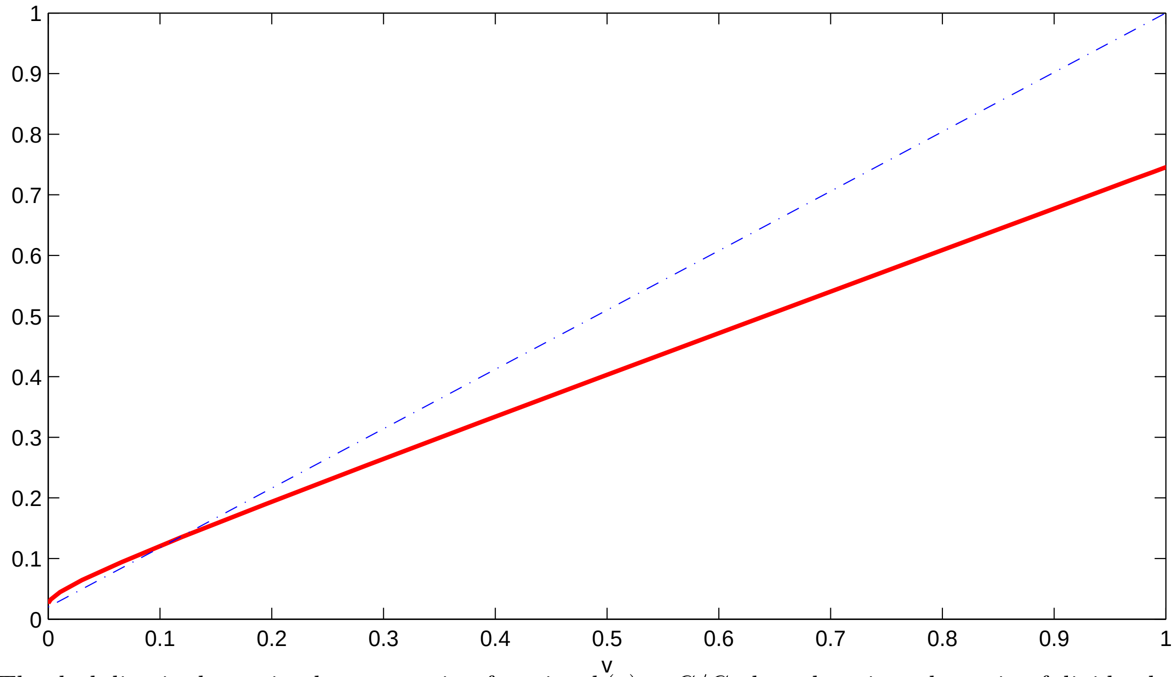
In fact, v is perfectly positively correlated with $\widetilde{Q}$: the economy becomes relatively less liquid when dividends grow faster than stored wealth. This is easy to see by rewriting Equation (1) as

$$v_{t+1}^{-1} = 1 + v_t^{-1}(1 - h(v_t))\widetilde{Q}_{t+1}^{-1}$$

and recalling $(1 - h(v)) \geq 0$. One could thus imagine a transition to a very illiquid state arising from a windfall dividend or from an unexpected deterioration in buffer stocks. While the latter corresponds more closely to how one usually thinks of a "liquidity shock", the two forces are symmetrical in their effect on v.

The excess returns, $\widetilde{Q}$, thus raise v, $h(v)$, and G, which unambiguously implies rising consumption. The precise sensitivity of C to $\widetilde{Q}$ shocks also varies with v. This variation is key to understanding the economy's behavior. With a little manipulation, the elasticity

Figure 1: *Optimal Consumption*



The dark line is the optimal consumption function $h(v) \equiv C/G$ plotted against the ratio of dividends to total available goods $v \equiv D/G$. Also shown is certainty-equivalent income as a fraction of G plotted as a dashed line. All parameter settings are as in Table 1.

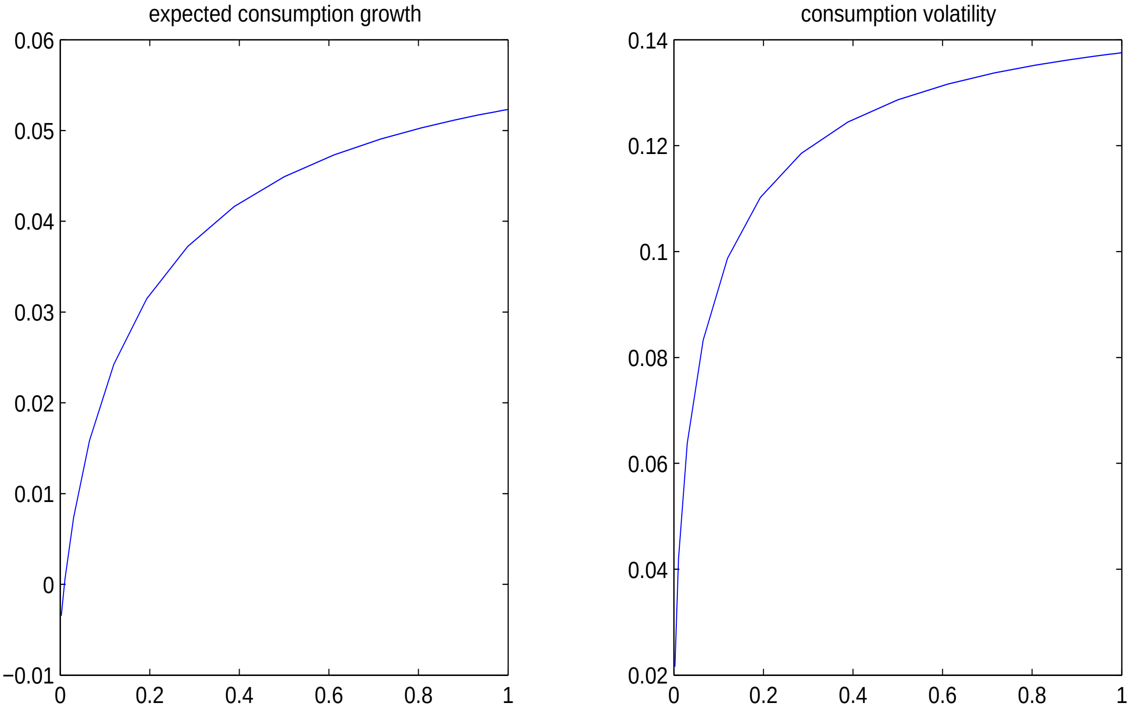
of consumption with respect to these shocks works out to be

$$\frac{\tilde{Q}}{C}\frac{\partial V}{\partial \tilde{Q}} = v + v(1-v)\frac{h'(v)}{h(v)}.$$

The concavity of the function h makes this elasticity an increasing function of v, rising rapidly towards unity. This pattern, in fact, shows precisely the sense in which liquid holdings act as a buffer stock. When the holdings are small (v is large), shocks to income translate directly into shocks to consumption. Even though consumption is a smaller fraction of income (since the agent wants to save to rebuild the buffer), it is more exposed to income.

With a given solution for $h(v)$, one can readily compute the exact moments of consumption changes at each point in the state space. Figure 2 shows the mean and volatility for the baseline parameter values used above. In line with the preceding discussion, both moments are initially dampened (for low v) but rise rapidly as the exposure of consumption to dividends rises. The model thus predicts significant time-variation in consumption growth and consumption risk as the real liquidity of the economy varies.

Figure 2: *Consumption Moments*

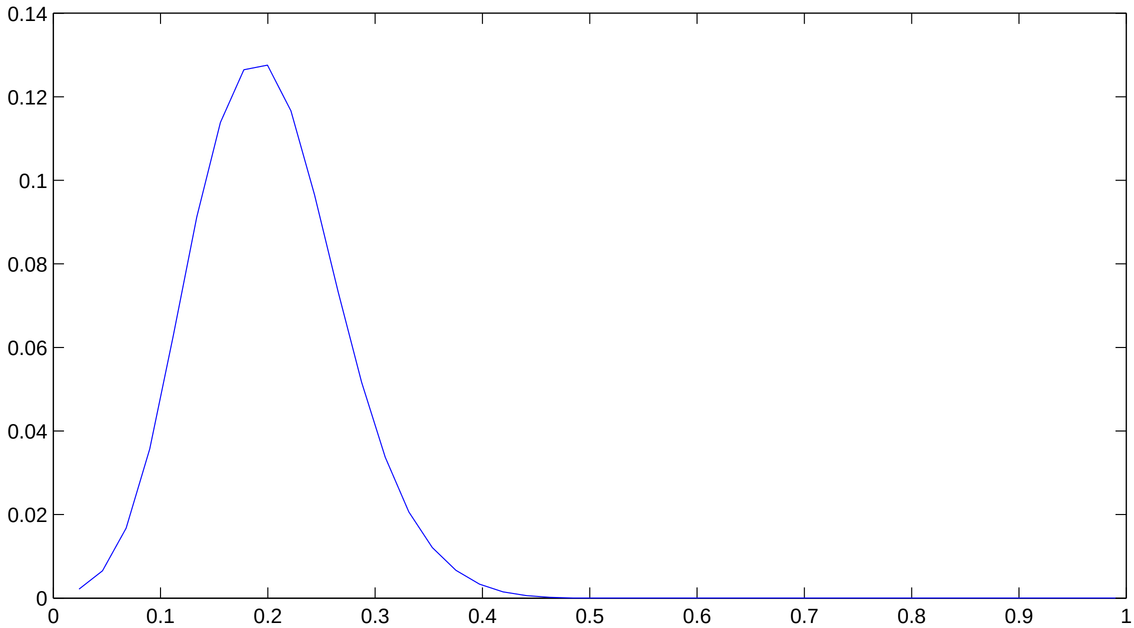


The left panel plots the conditional mean of log consumption growth, and the right hand panel plots the conditional standard deviation. The horizontal axis is v, the dividends-to-liquid balances ratio. All parameter settings are as in Table 1.

How much does that liquidity vary? Figure 3 shows the unconditional distribution of v, calculated by time-series simulation. Its mean and standard deviation are 0.204 and 0.065 respectively, implying a plus-or-minus one standard deviation interval of (0.139, 0.269). Using this distribution to integrate the conditional consumption moments, the unconditional mean and volatility of the consumption process are 0.031 and 0.110. In terms of dynamic variation, the standard deviation of the conditional mean and volatility are 0.0051 and 0.0079, respectively.

While analytical results on the the main quantities are not available, I numerically solve a range of alternative parameterizations to check the robustness of the important properties. The exact parameters were chosen to keep the economies stationary. This facilitates comparison by providing natural reference values of v for each economy. Table 2 reports fractiles of the steady state distribution of v for each case. (The caption gives the specific parameters for each one.) Table 3 then shows the ranges of $h(v)$, and of consumption volatility, $\sigma_C(v)$, for the cases. As expected, both the marginal propensities and consumption volatilities are increasing in all cases. These properties derive from the

Figure 3: *Unconditional Distribution of Dividend-Liquid Wealth Ratio*



The figure shows the unconditional distribution of v, the dividend-to-goods ratio, as computed from 40,000 realizations of a time-series simulation. The first 500 observations are discarded and a Gaussian kernel smoother has been applied. All parameter settings are as in Table 1.

basic underlying mechanisms described above, and, for that reason, further (unreported) numerical experimentation shows they apply generally to nonstationary parameterizations as well.

To recap, this section has shown some important, basic properties of consumption in this model. The propensity to consume current goods, $h()$, is increasing in the liquidity ratio, v, the percentage of current wealth coming from dividends. Subject to some parameter restrictions, this percentage is stationary, with a degree of mean-reversion determined by μ, r, ϕ, and γ. The consumption ratio, h, and consumption growth are then also stationary with the same characteristic time-scale. Remarkably, although the exogenous environment is *i.i.d.*, states with high and low levels of liquidity (or accumulated savings) seem very different. When liquidity is low, consumption is more volatile and income is saved; when liquidity is high, agents dissave and consumption growth is smoother. These dynamics lead to the main intuition needed to understand asset pricing in this economy.

Table 2: *Stationary Distributions: Alternative Parameterizations*

	Fractiles of illiquidity ratio				
	v_{10}	v_{25}	v_{50}	v_{75}	v_{90}
High impatience	0.2821	0.3315	0.3867	0.4416	0.4902
Low impatience	0.0525	0.0773	0.1114	0.1503	0.1899
High risk aversion	0.0245	0.0379	0.0570	0.0803	0.1061
Low risk aversion	0.0556	0.0819	0.1169	0.1584	0.1993
High growth	0.3778	0.4331	0.4919	0.5458	0.5916
Low growth	0.0052	0.0133	0.0270	0.0456	0.0670
High volatility	0.0006	0.0039	0.0148	0.0366	0.0640
Low volatility	0.2485	0.2933	0.3449	0.3977	0.4427

Model solutions for eight stationary cases are computed numerically and the unconditional distribution of the illiquidity ratio v is generated via simulation. The table reports the 10th, 25th, 50th, 75th, and 90th percentiles of these distributions. The first two lines use $\gamma = 6, \sigma = .14, \mu = .04$. (The "high" case puts $r = -.01, \phi = .10$. The "low" case puts $r = .05, \phi = 0$.) The third and fourth lines use $r = .02, \phi = .02, \mu = .03, \sigma = 0.14$. (The "high" case puts $\gamma = 8$. The "low" case puts $\gamma = 4$.) The fifth and six lines use $r = .02, \phi = .02, \gamma = 6, \sigma = 0.14$. (The "high" case puts $\mu = .06$. The "low" case puts $\mu = .02$.) The last two lines use $r = .02, \phi = .02, \gamma = 6, \mu = 0.03$. (The "high" case puts $\sigma = .20$. The "low" case puts $\sigma = .10$.)

3. Liquid Wealth and Asset Prices

Having solved the model to characterize consumption, one can immediately compute the price, expected return, and volatility of a claim to the dividend stream of the non-transformable endowment asset. Following Lucas (1978), the such a claim is being interpreted as the stock market. (The price of a claim to a unit of the transformable asset is identically one, of course, since it can be costlessly converted to current consumption.) This section outlines the moment computations and provides numerical illustrations. Under a broad range of parameter values, the first two moments of returns are monotonically increasing in the degree of liquid wealth. While analytical results are not available, I provide the economic logic underpinning these findings.

The computation starts by solving for the price-dividend ratio $g = g(v) \equiv P/D$, from investors' first-order condition. With the notation of the last section, this condition is

$$g(v_t) = \beta \mathrm{E_t} \left[\left(\frac{v_t h(v_{t+1})}{v_{t+1} h(v_t)} \widetilde{R}_{t+1} \right)^{-\gamma} \; [1 + g(v_{t+1})] \; \widetilde{R}_{t+1} \right]$$

Table 3: *Solution Properties*

	Panel I: fractiles of C/G				
	$h(v_{10})$	$h(v_{25})$	$h(v_{50})$	$h(v_{75})$	$h(v_{90})$
High impatience	0.2798	0.3193	0.3643	0.4088	0.4490
Low impatience	0.0902	0.1084	0.1320	0.1582	0.1842
High risk aversion	0.0416	0.0493	0.0594	0.0711	0.0834
Low risk aversion	0.0745	0.0944	0.1197	0.1492	0.1778
High growth	0.3823	0.4289	0.4784	0.5236	0.5618
Low growth	0.0262	0.0323	0.0408	0.0511	0.0620
High volatility	0.0212	0.0247	0.0314	0.0412	0.0514
Low volatility	0.2595	0.2985	0.3434	0.3893	0.4283
	Panel II: fractiles of consumption volatility				
	$\sigma_C(v_{10})$	$\sigma_C(v_{25})$	$\sigma_C(v_{50})$	$\sigma_C(v_{75})$	$\sigma_C(v_{90})$
High impatience	0.1231	0.1256	0.1279	0.1298	0.1311
Low impatience	0.0667	0.0776	0.0881	0.0966	0.1030
High risk aversion	0.0534	0.0633	0.0736	0.0824	0.0896
Low risk aversion	0.0841	0.0940	0.1028	0.1098	0.1147
High growth	0.1250	0.1273	0.1292	0.1307	0.1318
Low growth	0.0244	0.0410	0.0563	0.0690	0.0791
High volatility	0.0148	0.0257	0.0519	0.0767	0.0953
Low volatility	0.0879	0.0898	0.0915	0.0929	0.0939

The table reports properties of optimal consumption for the stationary cases described in Table 2. The first panel shows the consumption of available goods evaluated at the 10th, 25th, 50th, 75th, and 90th percentiles of the stationary distribution of the illiquidity ratio v. The second panel shows the volatility of log consumption growth at the same values.

or

$$g(v_t) = \beta R^{1-\gamma} \mathrm{E}_t \left[\left(\frac{v_t h(v_{t+1})}{v_{t+1} h(v_t)} \right)^{-\gamma} \; [1 + g(v_{t+1})] \; \widetilde{Q}_{t+1}^{1-\gamma} \right].$$

The excess dividend innovation, $\widetilde{Q}_{t+1}$, is the only source of uncertainty that affects the right-hand side since changes to the illiquidity ratio, v, are driven by the same shocks, via

$$v_{t+1} = \frac{v_t \widetilde{Q}_{t+1}}{(1 - h(v_t)) + v_t \widetilde{Q}_{t+1}}.$$

The function $g(v)$ can be found by iterating this mapping on the unit interval.[8] Once g is obtained, the distribution of returns to the claim can be evaluated from

$$\frac{(P_{t+1} + D_{t+1})}{P_t} = \widetilde{R}_{t+1}\ [1 + g(v_{t+1})]/g(v_t).$$

I subtract the log riskless rate from the log of these returns, and compute the first two moments of the resulting distribution.[9]

Since closed-form expressions are again not attainable, it is important to highlight the intuition behind the numerical results. This is straightforward. Having seen that consumption is less volatile when liquid balances are high, one can immediately infer that discount rates will be lower in these states since marginal utility is smoother and hence the economy is less risky. Further, when liquid balances are high, dividends comprise a smaller component of consumption. Thus a claim to the dividend stream has less fundamental risk when G is high relative to D. This implies that prices will be higher and risk premia lower when the economy is more liquid.

Figure 4 plots both the price-dividend ratio, g, and the price-available wealth ratio, f, for the parameter values in Table 1. The first function affirms the intuition above that the dividend claim must be more valuable when v is lower. In fact, g becomes unbounded as v approaches zero. This is not troublesome, however, because it increases slower than $1/v = G/D$. The plot of $f(v) = vg(v)$ goes to zero at the origin, indicating that the total value of the equity claim is not explosive. Indeed, f is monotonically increasing in v, which lends support to the interpretation of v as measuring the illiquidity of the economy, since this function is the ratio of the *value* of the non-transformable asset to the value of the transformable one.

In light of the lack of closed-form results, it is worth mentioning here the features of the pricing function that will matter when analyzing market resilience. Referring again to the figure, the fact that g explodes while f does not essentially bounds the convexity

[8] It is actually simpler to solve for the price-goods ratio $f(v) \equiv P/G = vg(v)$ which is not singular at the origin.

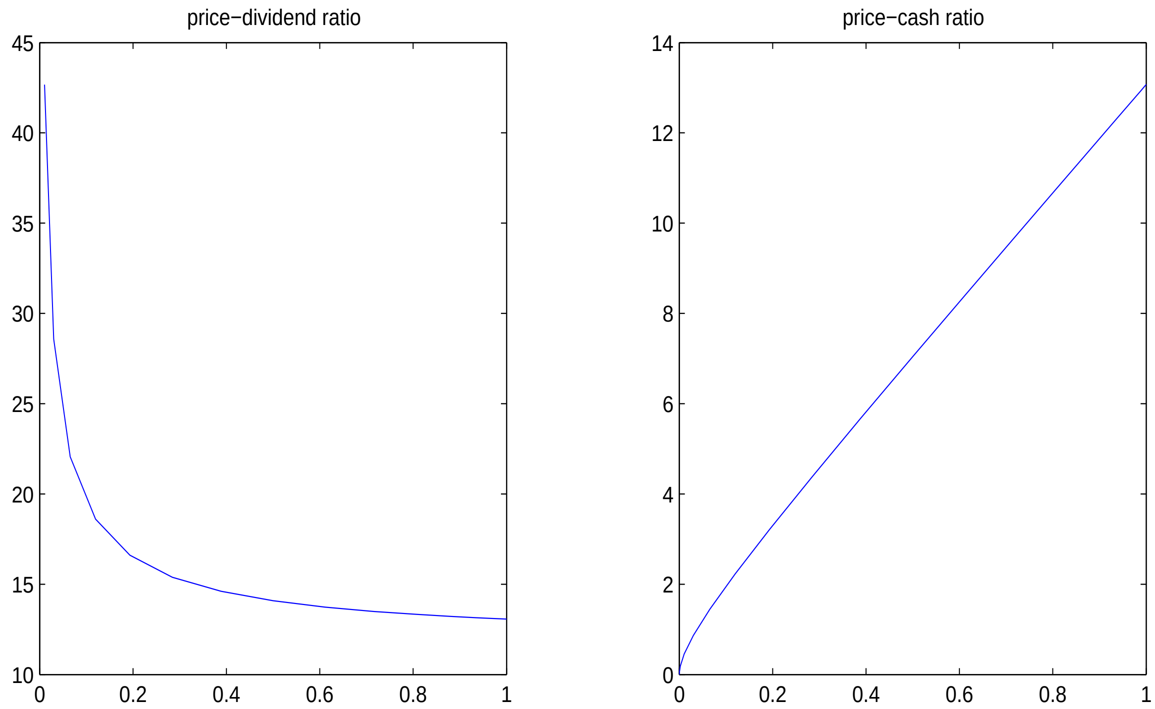
[9] When the liquid returns, $\widehat{R}$ are not constant, the riskless rate is found from the first order condition for the price of a one-period bond:

$$1/R^f = \beta R^{1-\gamma}\mathrm{E_t}\left[\left(\frac{h(v_{t+1})}{h(v_t)}[(1-h(v_t)) + v_t\widetilde{Q}_{t+1}]\right)^{-\gamma}\ \widehat{R}_{t+1}^{-\gamma}\right] = e^{\widehat{mu}-\gamma\widehat{\sigma}^2}.$$

Note that now $\log R^f \neq \log R \neq \log \mathrm{E}\widehat{R} \neq \mathrm{E}\log \widehat{R}$. However the important point is that all of these quantities are constant, and do not vary with the state of the economy.

of g to be no greater than that of v^{-1} in the neighborhood of the origin. While not visually apparent, a similar convexity bound holds on the entire unit interval.[10] While the generality of this property is conjectural, numerical experimentation suggests that it is robust. The curvature of the asset pricing function as a function of v is the key determinant of how much exogenous shocks to asset supplies will affect prices. As will be shown below in Section 4, that is tantamount to determining the liquidity of the securities market.

Figure 4: *Asset Pricing Functions*

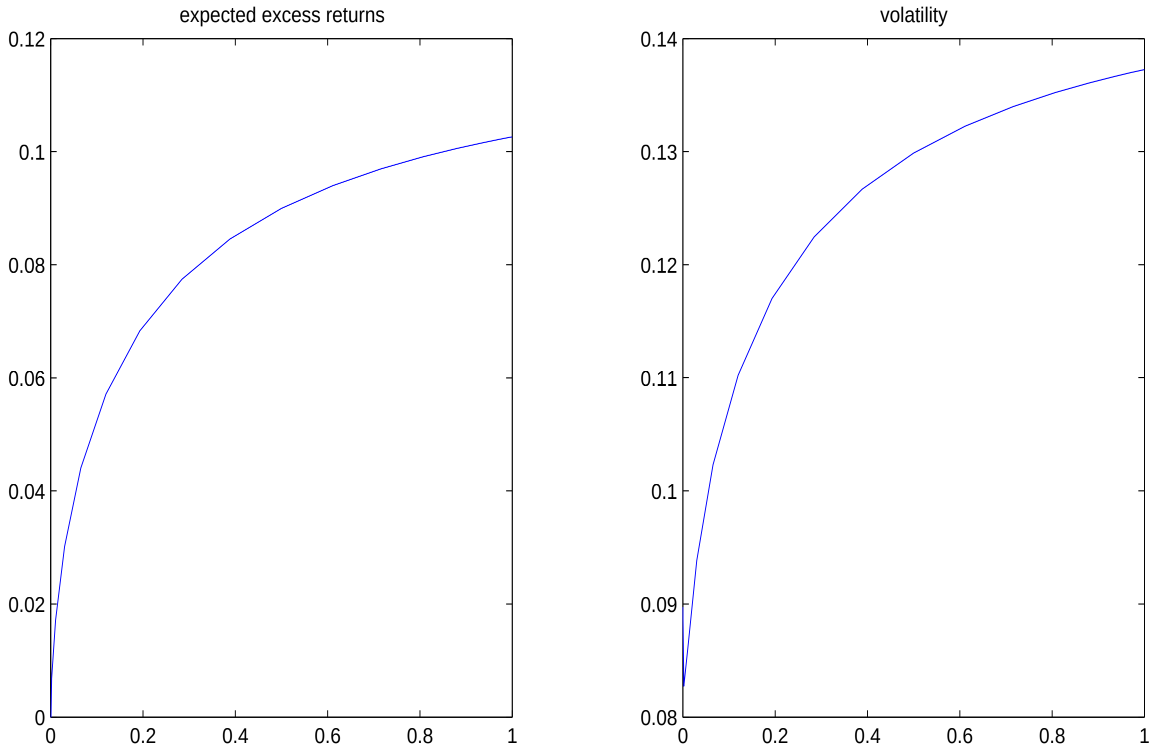


The left panel plots the ratio $g \equiv P/D$, and the right hand panel plots P/G. The horizontal axis is v, the dividends-to-liquid balances ratio. All parameter settings are as in Table 1.

Turning to return moments, Figure 5 evaluates the risk premium and volatility for the endowment asset as a function of v for the same parameter values. The plot verifies the intuition about risk premia. Volatility dynamics are also straightforward: while dividend innovations are homoscedastic, the volatility of marginal utility tracks that of consumption. As seen in the last section, consumption volatility is low when buffer stock savings are high.

[10] Technically, the requirement is $g'' \leq 2(g')^2/g$ which holds for functions of the form $Av^{-\alpha}$ as long as $\alpha \leq 1$.

Figure 5: *Asset Return Moments*



The left panel plots the conditional mean of continuously compounded excess returns, and the right hand panel plots their conditional standard deviation. The horizontal axis is v, the dividends-to-liquid balances ratio. All parameter settings are as in Table 1.

Table 4 verifies the increasing properties of return volatility and the decreasing property of the price-dividend ratio for the range of alternative parameter configurations described in Table 2. As before, the table evaluates these functions at fractiles of the unconditional distribution of the illiquidity ratio. This provides a natural gauge of how much variation in the moments would actually be experienced in a realization of each economy.

The results here reveal that the model provides a rich set of predictions about time variation in return dynamics. The theory implies that the economy undergoes cycles in real liquidity which affect the stock market, even as the capital stock of the endowment asset is held fixed. The real liquidity of the economy thus plays a role analogous to the "supply of credit" in models of intermediation capital. Even though there are no intermediaries and, indeed, no credit in the model, still asset prices rise and fall with the amount of available capital. As remarked in the discussion of consumption dynamics, even though the external environment in the model does not change (the exogenous shocks are *i.i.d.*), states of high and low real liquidity appear very different, with the

Table 4: *Asset Pricing Properties*

	Panel I: fractiles of P/D				
	$g(v_{10})$	$g(v_{25})$	$g(v_{50})$	$g(v_{75})$	$g(v_{90})$
High impatience	21.87	21.20	20.62	20.17	19.85
Low impatience	16.22	14.71	13.47	12.63	12.05
High risk aversion	25.40	21.88	19.03	17.15	15.81
Low risk aversion	29.61	26.86	24.77	23.34	22.42
High growth	16.42	16.04	15.72	15.48	15.31
Low growth	43.81	32.11	25.42	21.78	19.58
High volatility	74.29	40.77	24.24	17.20	14.11
Low volatility	24.56	24.04	23.58	23.23	22.98
	Panel II: fractiles of return volatility				
	$\sigma_P(v_{10})$	$\sigma_P(v_{25})$	$\sigma_P(v_{50})$	$\sigma_P(v_{75})$	$\sigma_P(v_{90})$
High impatience	0.1203	0.1229	0.1253	0.1273	0.1289
Low impatience	0.1071	0.1103	0.1138	0.1169	0.1194
High risk aversion	0.0928	0.0966	0.1006	0.1044	0.1079
Low risk aversion	0.1055	0.1102	0.1148	0.1188	0.1217
High growth	0.1249	0.1269	0.1287	0.1302	0.1312
Low growth	0.0976	0.0969	0.1001	0.1038	0.1075
High volatility	0.1457	0.1294	0.1274	0.1345	0.1413
Low volatility	0.0904	0.0918	0.0931	0.0942	0.0949

The table reports properties of claims to the endowment asset for the stationary cases described in Table 2. The first panel shows the price-dividend ratio evaluated at the 10th, 25th, 50th, 75th, and 90th percentiles of the stationary distribution of the illiquidity ratio v. The second panel shows the volatility of log returns at the same values.

latter being characterized by low prices, high volatility, and high expected returns.

Having shown that the level and volatility of stock prices can be affected by the supply of deployable capital, we now return to the topic of the resilience of the stock market. Are securities markets more liquid when the economy is more liquid? If so, why?

4. Changes in Market Liquidity

In what sense can financial claims in this paper's economy be said to be illiquid? After all, no actual trade in such claims takes place in the model, and, if it did, there are no frictions to make transactions costly.

Nevertheless, these observations do not mean that the market's demand curve for shares of the endowment asset is flat. In fact, in general, this will not be the case: marginal perturbations to a representative agent's portfolio will marginally alter his discount rates, altering prices. This paper uses the magnitude of this price effect – essentially the slope of the representative agent's demand curve – as the definition of a claim's secondary market illiquidity. It measures the price impact function that would be faced by an investor who did wish to trade with the market (i.e. with the representative agent) for whatever reason. Likewise, it measures the willingness of a typical investor (who has the holdings and preferences of the representative agent) to accommodate small perturbations to his portfolio.

The notion of market illiquidity as the slope of the aggregate demand function for shares of stock is due to Pagano (1989).[11] The formal definition used here is from Johnson (2006). Taking the value and price functions of the representative agent to be functions of his holdings, $X^{(0)}$ and $X^{(1)}$, of any two assets, the illiquidity of asset one with respect to asset zero is computed as the change in the agent's marginal valuation of asset one following a value-neutral exchange for units of asset two:

Definition 4.1 *The illiquidity $\mathcal{I} = \mathcal{I}^{(1,0)}$ of asset one with respect to asset zero is the elasticity*

$$\mathcal{I} = -\frac{X^{(1)}}{P^{(1)}} \frac{dP^{(1)}(\Theta(X^{(1)}), X^{(1)})}{dX^{(1)}} = -\frac{X^{(1)}}{P^{(1)}} \left(\frac{\partial P^{(1)}}{\partial X^{(1)}} - \frac{P^{(1)}}{P^{(0)}} \frac{\partial P^{(1)}}{\partial X^{(0)}} \right), \tag{1}$$

where $(\Theta(x), x)$ is the locus of endowment pairs satisfying $J(\Theta(x), x) = J(X^{(0)}, X^{(1)})$.

The definition stipulates that the derivative be computed along isoquants of the value function (parameterized by the curve Θ) and the second equality follows from the observation that value neutrality implies

$$\frac{d\Theta(X^{(1)})}{dX^{(1)}} = \frac{dX^{(0)}}{dX^{(1)}} = -\frac{P^{(1)}}{P^{(0)}}.$$

[11] There is, of course, a long history in the microstructure literature of measuring liquidity by the slope of the demand curve of an *ad hoc* market maker.

While the definition depends on the choice of asset zero, in many contexts there is an asset which it is natural to consider as the medium of exchange. In the model of Section 2 above, the storable asset is the obvious unit since its relative price in terms of goods is clearly constant, $P^{(0)} = 1$.[12]

In Johnson (2006), $\mathcal{I}$ is computed only in endowment economies. In that case, the perturbation compares prices in economies with differing, but fixed, asset supplies. In the present case, the concept is extended to economies in which quantities are not fixed. In particular, the perturbation exercise now envisions an exchange of endowment shares for transformable capital. While, mechanically, this is still just a straightforward differentiation, conceptually there is an extra repercussion, as the exchange now leads to an endogenous decision about what is done with the consumable goods. Changing relative supplies (at the beginning of the period) changes consumption. The representative agent will save or consume additional goods to different degrees in different circumstances.

Note that, like prices themselves, the elasticity $\mathcal{I}$ can be computed whether or not trade actually takes place in equilibrium. If the economy were disaggregated, and some subset of agents experienced idiosyncratic demand shocks forcing them to trade, they would incur costs proportional to $\mathcal{I}$ as prices moved away from them for each additional share demanded. (An application which demonstrates the equivalence of individual price impact and $\mathcal{I}$ in a setting with explicit trade appears in Johnson (2007).) Equivalently, $\mathcal{I}$ is proportional to the percentage bid/ask spread (scaled by trade size) that would be quoted by competitive agents in the economy were they required to make two-way prices. Thus, it captures two familiar notions of illiquidity from the microstructure literature.

Two simple examples may help illustrate the concept, and also clarify the analysis of illiquidity in the full model.

First, consider pricing a claim at time t to an asset whose sole payoff is at time T in a discrete-time CRRA economy. As usual, $P_t^{(1)} = \mathrm{E_t}\left[\beta^{T-t} u'(C_T) D_T / u'(C_t)\right]$. Further suppose the asset is the sole source of time-T consumption: $C_T = D_T = D_T^{(1)} X^{(1)}$, and let the numeraire asset be any other claim not paying off at T or t. Then, differentiating,

$$\frac{dP_t^{(1)}}{dX^{(1)}} = \frac{1}{u'(C_t)} \mathrm{E_t}\left[\beta^{T-t} u''(C_T)(D_T^{(1)})^2\right] = -\gamma \mathrm{E_t}\left[\beta^{T-t} \frac{u'(C_T)}{u'(C_t)} \frac{D_T^{(1)}}{X^{(1)}}\right] = -\gamma \frac{P^{(1)}}{X^{(1)}}$$

or $\mathcal{I} = \gamma$. In this case, the effect of asking the agent to substitute away from time-

[12]Technically, one should distinguish between the quantity of claims to a unit of the physical asset and the capital stock, G, of that asset. But since the exchange rate is technologically fixed at unity, I make no distinctions below and use G and $X^{(0)}$ interchangeably.

T consumption causes him to raise his marginal valuation of such consumption by the percentage γ, which is also the inverse elasticity of intertemporal substitution under CRRA preferences. If this elasticity were infinite, the claim would be perfectly liquid. Notice that the effect is not about risk bearing: no assumption is made in the calculation about the risk characteristics of the *other* asset involved. So the exchange could either increase or decrease the total risk of the portfolio.

Now consider a similar exchange of asset one for units of the consumption good. Then, in the computation of $\mathcal{I}$, there is an extra term in $dP^{(1)}/dX^{(1)}$ which is

$$\mathrm{E_t}\left[\beta^{T-t}u'(C_T)D_T\right]\ \frac{d}{dX^{(1)}}\left(\frac{1}{u'(C_t)}\right) = -P^{(1)}\,\frac{u''(C_t)}{u'(C_t)}\,\frac{dC_t}{dX^{(1)}} = \gamma\frac{P^{(1)}}{C_t}\,\frac{dC_t}{dX^{(1)}}.$$

Now the value neutrality condition implies $dC_t/dX^{(1)} = -P^{(1)}$ so that $\mathcal{I}$ becomes

$$\gamma\left[1+\frac{P^{(1)}X^{(1)}}{C_t}\right].$$

It is easy to show that this is the illiquidity with respect to consumption for a general (i.e. not just one-period) consumption claim as well. Here the intertemporal substitution effect is amplified by a (non-negative) term equal to the percentage impact of the exchange on current consumption: $P^{(1)}X^{(1)}/C_t = \left|(X^{(1)}/C_t)\ dC_t/dX^{(1)}\right|$. This term may be either large or small depending on the relative value of future consumption. In a pure endowment economy with lognormal dividends and log utility, for example, current consumption is $C_t = X^{(1)}D_t^{(1)}$ and the extra term is price dividend ratio, which is $1/(1-\beta)$, which would be big. The intuition for this term is that marginally reducing current consumption (in exchange for shares) raises current marginal utility. So, if the representative agent is required to purchase $\Delta X^{(1)}$ shares and forego current consumption of $P^{(1)}\Delta X^{(1)}$, his discount rate rises (he wants to borrow) and he would pay strictly *less* than $P^{(1)}$ for the *next* $\Delta X^{(1)}$ shares offered to him.

In what follows, it will be useful to think of the mechanism in these examples as two separate liquidity effects. I will refer to that of the first example, captured by the term $\gamma\cdot 1$, as the future consumption effect, and that of the second, captured by $\gamma\cdot P^{(1)}X^{(1)}/C_t$, as the current consumption effect.

Returning to the model of Section 2, such explicit forms of $\mathcal{I}$ in terms of primitives are not available. (Direct differentiation of the discounted sum of future dividends is intractable because future consumption depends in a complicated way on the current endowments.) However it is simple to express $\mathcal{I}$ in terms of the functions g (the price-dividend ratio) and f (the value ratio), which are both functions of v, the dividend-liquid balances ratio.

Proposition 4.1 *In the model described in Section 2,*

$$\mathcal{I} = -v(1 + vg(v))\frac{g'(v)}{g(v)} = (1 - v\frac{f'(v)}{f(v)})(1 + f(v)).$$

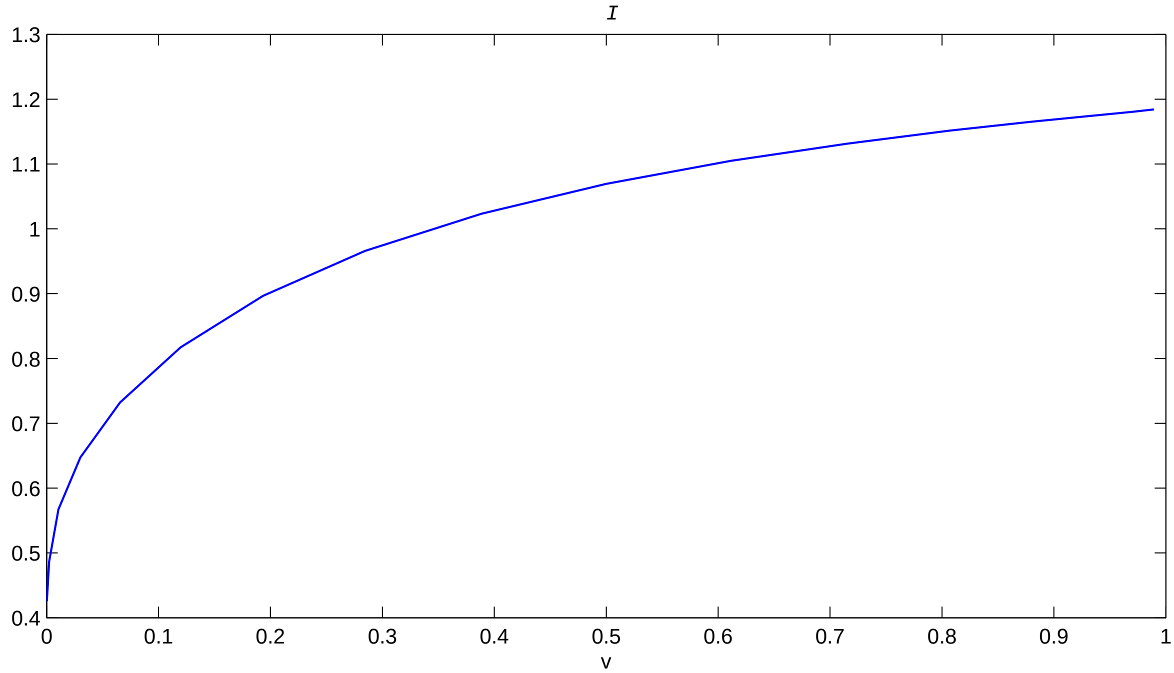
Illiquidity is positive in this economy because the price-dividend ratio is a declining function of v. Adding shares in exchange for consumable goods mechanically shifts v to the right. As discussed above, when shares make up a larger fraction of the consumption stream their fundamental risk increases and their value declines.

Figure 6 plots illiquidity using the parameter values from Table 1. Notice first the most basic features, the level and variation of the function. The magnitude of illiquidity is both significant and economically reasonable. An elasticity of unity implies a one percent price impact for a trade of one percent of outstanding shares. This is the order of magnitude typically found in empirical studies of price pressure for stocks. Further, market liquidity is time-varying in this model. It is not a distinct state variable, of course, yet it is still risky in the sense of being subject to unpredictable shocks. While the current parameters restrict v, and hence $\mathcal{I}$, to a rather narrow range, even so, it is possible for illiquidity to more than double.

This brings us to the topic of how and why market liquidity changes here. The figure clearly provides the fundamental answer: illiquidity rises when v does. Or, to stress the main point, the stock market is more liquid when technologically liquid capital is in greater relative supply. This is the heart of the paper's results.

To understand why this occurs, consider the role that the availability of a savings technology plays in the determination of price impact. In effect, it dampens both of the illiquidity mechanisms in the pure-endowment examples above. When the liquidity provider (the representative agent) chooses to use some available capital to purchase additional shares – instead of being forced to forego consumption – current marginal utility does not rise as much. In addition, marginally depleting savings today raises the expected marginal utility of future income, which then raises the valuation of future dividends. Hence the impact of foregone consumption (illustrated in the earlier examples) on both the numerator and denominator of the marginal rate of substitution are buffered by the use of savings. But now recall from the last section that the propensity to save

Figure 6: *Stock Market Illiquidity*



The figure shows the elasticity $\mathcal{I}$ as a function of v for the model of Section 2. All parameter settings are as in Table 1.

when given an extra unit of G (or to dissave when required to give up a unit) falls with v. Because h rises with v (for the very general reasons discussed previously), discount rates are less affected by portfolio perturbations when v is low.

Appendix B analyzes the effects in more detail in the two-period case which corresponds to the previous examples. Even in this case, a formal proof that $\mathcal{I}$ is increasing is unobtainable. However it is possible to isolate the individual terms in $\mathcal{I}'$ and to see how each rises with the propensity to consume.

Mechanically, differentiating the expression in the proposition shows that $\mathcal{I}'$ will be positive as long as $g(v)$ is not too convex. (See note 10.) In fact, it is sufficient that $\log g$ is concave. And concavity is equivalent to the assertion that g'/g gets bigger (more negative) as v increases, meaning the percentage price impact of an increase in v is increasing.

As shown in Table 5, the increasing property of $\mathcal{I}$ holds in all the parameter cases considered in Table 3. The table reveals some cases – e.g., the low growth and high volatility parameterizations – in which the equilibrium variation can exceed a factor of four.

As with the properties of consumption and asset returns examined earlier, the mono-

Table 5: *Market Illiquidity: Alternative Cases*

	Illiquidity fractiles				
	$\mathcal{I}(v_{10})$	$\mathcal{I}(v_{25})$	$\mathcal{I}(v_{50})$	$\mathcal{I}(v_{75})$	$\mathcal{I}(v_{90})$
High impatience	1.3747	1.4238	1.4708	1.5115	1.5435
Low impatience	0.3669	0.4217	0.4751	0.5228	0.5622
High risk aversion	0.4179	0.4769	0.5353	0.5880	0.6335
Low risk aversion	0.5793	0.6491	0.7173	0.7785	0.8257
High growth	1.1990	1.2394	1.2773	1.3091	1.3338
Low growth	0.1080	0.2660	0.3602	0.4339	0.4910
High volatility	0.0879	0.1034	0.3168	0.4349	0.5108
Low volatility	0.8991	0.9356	0.9725	1.0029	1.0248

The table reports market illiquidity for the stationary cases described in Table 2 evaluated at the 10th, 25th, 50th, 75th, and 90th percentiles of the stationary distribution of the illiquidity ratio v.

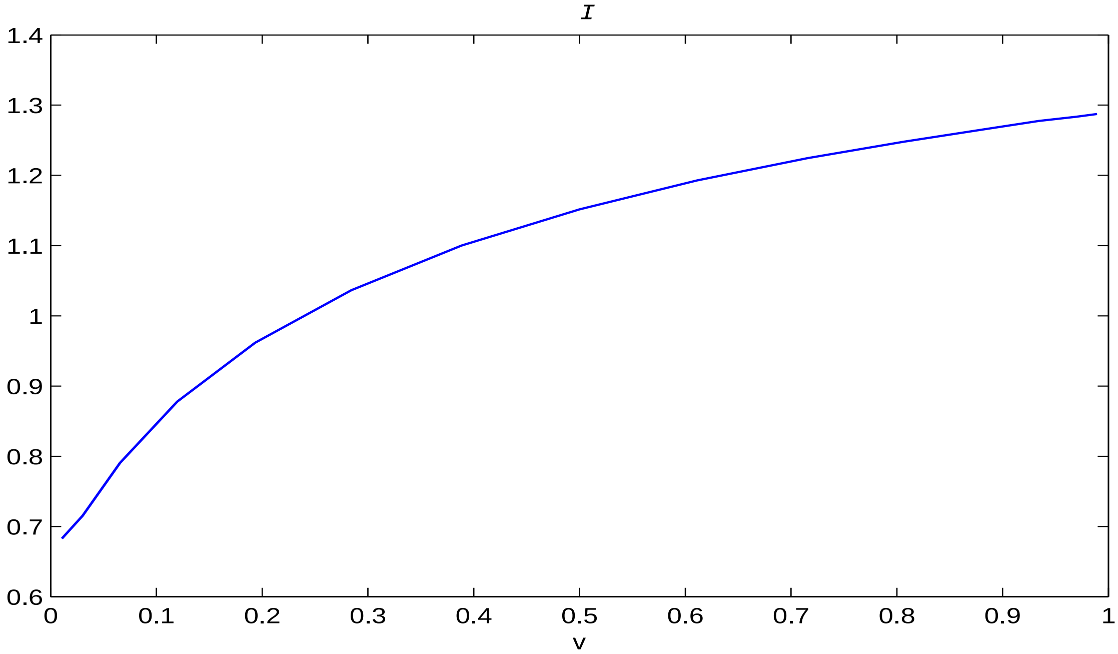
tonicity property here extends beyond the configurations examined in the table. Those cases enforce stationarity and also adopt the assumption that "liquidity shocks" ($\widehat{R}$) are independent of "excess returns" ($\widehat{Q}$). Neither condition is necessary.

Figure 7 shows market illiquidity for a parameter configuration in which $\widehat{R}$ and $\widetilde{R}$ are independent and have equal variance. In every important respect, including $\mathcal{I}' > 0$, this case behaves like the benchmark one. This shows that the key distinction in the model is not between "safe" and "risky" assets. The illiquidity of the endowment shares is not rising with v because of any change in risk of the economy, but because its capital stock becomes increasingly nontransformable.

Figure 8 shows illiquidity for a nonstationary case. This version has insufficient risk aversion to induce agents to save, even as their liquid capital dwindles. There is a dramatic deterioration in market liquidity with v for this economy, with $\mathcal{I}(v)$ rising by over an order of magnitude between $v = 0.1$ and $v = 0.9$, suggestive of a collapse in secondary market conditions or extreme financial fragility.

The above two cases also share the dynamic features described in Sections 2 and 3: as the liquid balances ratio increases, stocks become cheap (the price dividend ratio falls) and risk premia and risk both go up. Because v determines all dynamic quantities in the model, the covariances of market liquidity immediately follow from the positive relation between $\mathcal{I}$ and v. In particular, and in line with numerous empirical studies, market liquidity falls as the market does, and as volatility rises.

Figure 7: *Stock Market Illiquidity: Equal Variance Case*



The figure shows the illiquidity, $\mathcal{I}$, as a function of v using $\gamma = 6, \phi = 0.05, \widehat{\mu} = 0.05, \widetilde{\mu} = 0.02, \widehat{\sigma} = \widetilde{\sigma} = 0.10$, and $\rho = 0$.

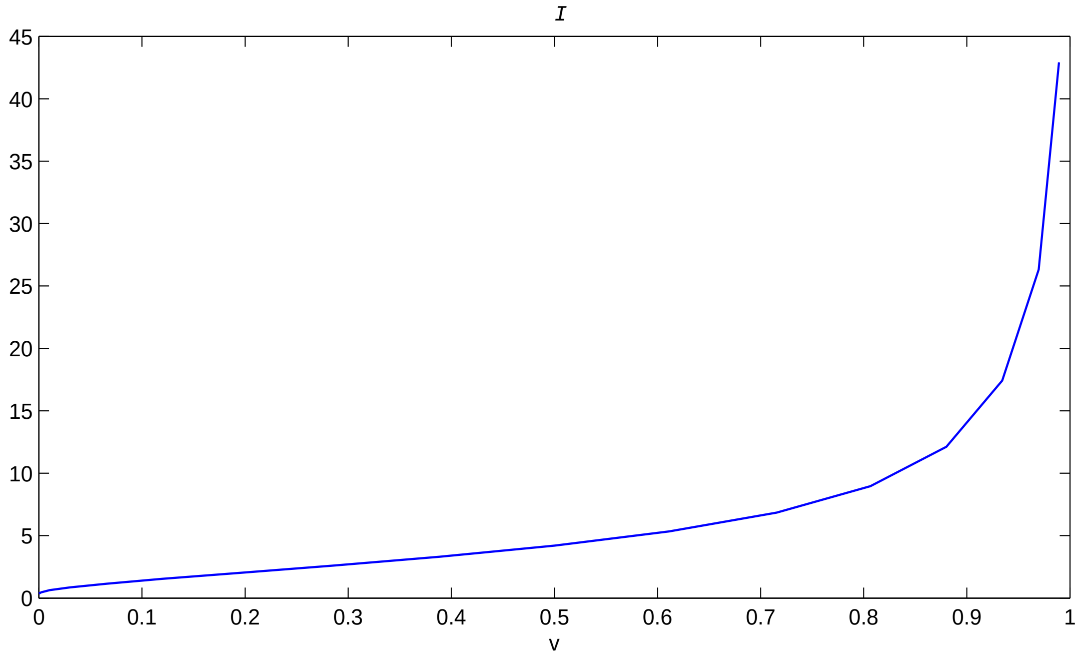
Summarizing, markets are illiquid in this economy ($\mathcal{I}(v) > 0$) because discount rates rise with the proportion of non-transformable endowment asset holdings. Markets are *increasingly* illiquid as liquid balances decline ($\mathcal{I}'(v) > 0$) because this impact on discount rates itself rises. Discount rates are affected more strongly by portfolio perturbations when these balances are low because consumption – not savings – absorbs a higher percentage of the adjustment.

5. Interpretation

5.1. Identifying Liquid Capital

Interpretation and assessment of the model's implications clearly depend on what the stylized quantities in its economy are taken to represent. Dividends in the model, for example, are also the only source of income aside from stored wealth, and so, for some purposes, could be understood to include labor income. However, when assessing properties of claims to D, the original interpretation as equity dividends is appropriate, since

Figure 8: *Stock Market Illiquidity: Nonstationary Case*



The figure shows the illiquidity, $\mathcal{I}$, as a function of v using $\gamma = 2, \phi = 0.02, r = 0.02, \mu = 0.03$, and $\sigma = 0.10$.

claims to labor income are not traded. Similarly with available wealth G (or liquid capital balances, $K \equiv G - D$), several identifications suggest themselves.

In terms of a standard real business cycle model, K represents the capital stock committed to a particular technology defined by its low adjustment costs. The literature recognizes that adjustment costs differ across sectors. The present paper highlights some aggregate consequences of this cross sectional difference. What matters for the analysis is not the ease with which one type of capital can be converted to another, but the ease with which it can be converted to consumption. With this understanding, the natural candidates for empirical counterparts to the adjustable sector would be the distribution, warehousing, wholesaling and perhaps retail industries. These businesses have a high percentage of assets that can be liquidated directly for consumption purposes.

This identification seems unpromising, though, in the sense that it is unlikely that making adjustments to these sectors is really an important mechanism for aggregate buffer stock savings. On the other hand, households and businesses can and do adjust their real liquid wealth by increasing or decreasing their holdings of consumable goods. In fact, this is the most literal interpretation of K: commodity stockpiles and product inventories. These do not constitute a "sector", but do represent a significant component of capital (which would include most of the assets of the sectors mentioned above). Under this interpretation, the model implies a direct role for these inventories in influencing financial markets. Testing these implications poses challenging measurement issues because data on commodity and product inventories – particularly at the household level – are not generally collected or reported.

Households and businesses also, of course, use cash and money market instruments to adjust their available wealth. Cash and guaranteed deposits *are* technologically liquid in that they can be exchanged immediately for goods and services without first having to be sold in a secondary market. Financial assets are not real assets, however, and, while there is a large literature devoted to the asset pricing effects of "the supply of credit", the present model is not rich enough to incorporate real effects of financial claims. Still, if one looks through financial claims to the real investments underlying them, there may be a case for the monetary interpretation of K. Specifically, netting out the liquid financial claims across the holdings of (private sector) agents, one is left with the monetary base as a remainder. This suggests, at least, including narrow money as a component of liquid capital. Arguably, changes in the real value of the monetary base (total central bank liabilities) do represent changes in the economy's aggregate buffer stock.

If one further nets liquid claims across the government sector, the remainder is the net asset base, or reserves, of the central bank. This last quantity is a common gauge of a country's liquidity. Some practitioners, including central bankers, sum this value across countries to measure global liquidity.[13] While these monetary proxies miss the components of physically liquid capital described above, they are directly observable and may capture a common component of variation.

The essential hypothesis of the paper is that agents can, at least in the short run, save without increasing risky (or irreversible) physical investment. The model can only offer a reduced form depiction of how that process works. Whatever the appropriate measure of available, technologically liquid, capital is, it is hard to ignore the fact that the model delivers a depiction of the relationship between this quantity and the behavior of the stock market that closely parallels that of much more complicated models of credit, intermediary capital, and market frictions. Specifically, many such models (and conventional wisdom) ascribe the coincidence of lower markets, higher volatility, and greater market fragility to the amplifying affect of reduced intermediary capital and credit constraints. (And, in the other direction, an excess of liquid capital is thought to

[13]Netting seems perhaps more appropriate than summing. If one nets reserves across countries, the remainder is again a commodity inventory: central bank gold holdings.

lead to artificial price increases for similar reasons.) Without minimizing the importance of these channels, the model here provides an additional potential explanation for these linkages. The theory here also traces the interactions to the common effect of the supply of liquid capital. It does so by viewing that capital as a real quantity whose relative supply affects equilibrium consumption and discount rates.

5.2. Intervention

While the model contains neither an intermediary sector nor any government entity, it nevertheless can be extended to provide a consistent, quantitative treatment of policy actions designed to alter the real level of liquid balances. The treatment is stylized: both the mechanism and the motivation for such an intervention are unmodelled. But, given that these actions do occur, the theory may offer useful perspective in understanding how and why they work.

As an example, consider the economy whose market illiquidity is plotted in Figure 8. The economy is nonstationary: agents consume "too much" and exhaust their savings, Any observed history would feature a steady drift of v towards one. As we have seen, this would entail rising volatility, falling asset prices, and a sharp spike in market illiquidity. Even though the external uncertainty facing agents is the same in all states (dividend growth is *i.i.d.*), the economy inevitably approaches something that looks like a "liquidity crisis." I now describe the type of intervention that can be entertained without altering the construction of the equilibrium.

Let today be t and assume that at some $\tau > t$ a random process will dictate a positive quantity ΔG to be added to the representative agent's cash holdings, G_τ, in exchange for a number of shares $\Delta X^{(1)}$ of the endowment stream. The crucial assumption will be that these quantities are determined so that the agent is *indifferent* to the exchange, i.e. it leaves his value function unchanged. Such an intervention can be viewed as self-financing in the sense that it involves no transfer of value from the intervening entity to the economy. For example, this could capture a central bank conducting a competitive reverse-auction to purchase endowment shares.

Mechanically, such an intervention shifts the ratio v to the left, as the numerator decreases and the denominator increases.[14] From a comparative static point of view, this would increase the price-dividend ratio, lower the risk premium, and reduce the volatility of consumption and of the stock market.

[14] To be careful, the previous notation needs to be augmented to reflect the variable number of shares

Is this conclusion justified from a dynamic point of view? Or would rational anticipation of the intervention at τ alter the equilibrium at t, rendering the comparative statics invalid? As the following proposition shows, the analysis is actually robust to interventions quite generally.

Proposition 5.1 *Let $\{\tau_k\}_{k=1}^K$ be an increasing sequence of stopping times $t < \tau_1 \ldots \tau_K$, and let $\{\delta_k\}_{k=1}^K$ be a sequence of random variables on $\mathcal{R}^+$. Suppose that at each stopping time an amount $\Delta G_{\tau_k} = G_{\tau_k}\ (\delta_k - 1)$ of goods are added to the representative agent's holdings in exchange for an amount of shares $\Delta X_k^{(1)}$ that leaves his value function unchanged. (If no such quantity exists, no exchange takes place.) Then, the value function, $J = J(G, v)$, consumption function, $h(v)$, and pricing function, $P(G, v)$, at time t are identical functions to those in the economy with no interventions when the endowments are fixed at their time-t amounts.*

The proposition tells us that we can consistently augment any version of the model of Section 2 to include an arbitrarily specified policy rule for competitive purchases or sales, and ignore the effects of those future exchanges in computing prices and liquidity. The underlying logic is simple: since the agent knows the intervention won't alter his value function at the time it occurs, the *ex ante* probability distribution of future value functions is unchanged. Hence today's optimal policies are still optimal, regardless of the intervention, which means the value function today is unaltered. Note that the proposition does not rule out that the timing and amounts of the actions could depend on the state of the economy, such as the price of equity.[15]

Returning to the numerical example above, the proposition implies that the comparative static analysis is, in fact, dynamically consistent. The only effect of a value-neutral open-market operation is to re-set the current value of v. In fact, one could make this non-stationary version of the model effectively stationary by imagining periodic "rescues" by the authorities when v approaches some higher limit.

of the endowment claim (so far implicitly set to one). So write

$$v_t \equiv \frac{D_t}{G_t} = \frac{D_t^{(1)}\, X_t^{(1)}}{G_t}.$$

That is, the superscript will denote per-share quantities. Thus, also, $P^{(1)} = D^{(1)}\, g(v)$ will be the per-share price of an endowment claim. If the representative agent sells shares, then, his stream of dividends is lowered, which is what v measures. The per-share dynamics of the D process is not changed.

[15]There is an implicit assumption that the intervention does not alter agents' information sets by conveying information about future value of D.

Moreover, the model then immediately permits computation of the intervention price. For example, suppose when v hits $\bar{v} = 0.95$, an open-market operation is undertaken to lower it to $\underline{v} = 0.25$. Then we can solve the following system

$$\begin{aligned} \frac{\underline{v}}{\bar{v}} &= \frac{G}{X^{(1)}} \frac{X^{(1)} - \Delta X^{(1)}}{G + \Delta G} \\ J(G, \bar{v}) &= J(G + \Delta G, \underline{v}) \end{aligned}$$

for $\Delta G / \Delta X$, the average price.[16] The system yields the price as a fraction of current dividends.

In this example, the average intervention price works out to be 34.02 (times D), which compares to a marginal price of $g(.95) = 26.11$ at the time of intervention. The authority thus appears to "overpay" for the risky shares. However, the action itself sufficiently lowers discount rates so that the marginal price rises to $g(.25) = 38.01$ after intervention.

Incorporating intervention thus enriches the descriptive range of the model and provides a positive, testable theory of the effects of policy actions. Again, it is worth noting that, while the model's predictions seem to align with those of standard models of constrained intermediaries, the mechanism here is very different.

To an observer of the economy in this example, it could well appear that the deterioration of prices and increases in volatility occur *because* of the lack of market liquidity. The apparent success of the intervention could seem to support the idea that the scarcity of deployable capital caused a decline in intermediation, causing the rise in market illiquidity, and leading to the seemingly distressed state. Yet neither of the above inferences need follow from the observed linkages. Market illiquidity and risk premia may rise simultaneously without the former having anything to do with the latter. A decrease in liquid balances can cause both, but without operating through the constraints of intermediaries.

6. Conclusion

Understanding the fundamental factors driving market liquidity is a crucial issue for both investors and policy makers. This paper describes an economic mechanism linking the resilience of prices with real, technological liquidity, defined as the fraction of the

[16]By homogeneity, the value function can be written $J(G, v) = \frac{1}{1-\gamma} G^{1-\gamma} j(v)$. The function j can be computed immediately by iteration, given the optimal consumption function.

capital stock which is available for consumption. In the model, when available balances are low, agents are less willing to accommodate others' trade demands because doing so entails more adjustment to current consumption. This is a primitive economic effect, stemming from the properties of the equilibrium consumption function, and not from microstructure effects or credit market frictions. The work thus contributes to the evolving understanding of the dynamics of market liquidity and liquidity risk.

More broadly, the theory highlights the role of adjustable capital in affecting all aspects of asset markets. Liquid savings serve to buffer consumption from production shocks, leading to lower volatility, risk premia, and discount rates. The paper's model provides an explicit and tractable quantification of time-varying moments, linking them to a new state variable, whose importance has perhaps not been previously appreciated. Despite the model's sparse structure, numerical results show that variations in market liquidity and return moments can be large.

An important task for future research is to test for the presence of the mechanism driving the paper's results. The model's effects derive from an increasing marginal propensity to consume available goods when savings are low, which effectively means more sensitivity of consumption to income shocks. While there is a large literature assessing the predictions of the buffer stock savings model at the individual level, there is little direct evidence on the predictions applied at the aggregate level. Here, specifically, the question is whether low liquid savings leads to less consumption smoothing. Addressing this question requires identifying liquid savings, i.e., aggregate buffer stocks, as well as measuring changes in consumption smoothing over time.

The second question the argument raises is whether low consumption smoothing, in turn, entails more fragile secondary markets. Consumption-based asset pricing models face well known empirical problems, suggesting that the present paper's use of standard preferences and *i.i.d.* Gaussian shocks may be too restrictive. On the other hand, the asset market predictions here (including those on volatility and expected returns) point to changes in consumption risk (rather than its level) as potential drivers of changing market conditions.

Finally, cutting out the middle linkage in the last two conjectures, one could look for direct evidence on the association between levels of liquid savings and market liquidity. As discussed in Section 5.1, monetary variables may or may not be appropriate as proxies for the former. However quantifying the association between monetary variables and market resilience is of interest in its own right as it bears directly on the policy goal of maintaining stable markets. In monthly data from 1965 through 2001, Fujimoto

(2004) finds that several measures of aggregate market illiquidity are significantly lower during expansionary monetary regimes than during contractionary ones. Vector autoregressions in the same study indicate a significant response of illiquidity to innovations in nonborrowed reserves and the Federal Funds rate. In addition, Chordia, Sarkar, and Subrahmanyam (2005) report that bid/ask spreads in stock and bond markets were negatively correlated with measures of monetary easing during three high-stress periods from 1994 to 1998.

There is, then, empirical support for the connection between monetary liquidity and market liquidity. This is consistent with models of financially constrained, segmented market makers, and accords with the conventional understanding of practitioners. The present paper suggests that this interpretation may not be the whole story. The same association follows from the frictionless, equilibrium effect modelled here. Financial constraints and inefficient markets all may well contribute to the determination of market liquidity. The ideas are not mutually exclusive. However they may have very different implications about the role of institutions and the welfare effects of intervention.

Appendix

A. Proofs

This appendix collects proofs of the results in the text.

Proposition 2.1

Proof. This proof will restrict attention to policy solutions in the class of limits of solutions to the equivalent finite-horizon problem. So consider the finite-horizon problem with terminal date T. Let h_t denote the optimal consumption-to-goods ratio at time t.

Clearly h_t cannot exceed one, since this would lead to a positive probability of infinitely negative utility at T. The assumption of the proposition is then that, at each v, we have an interior solution for h_t (at least for $T-t$ sufficiently large). In that case, h_t must satisfy the first order condition

$$h_t^{-\gamma} = \beta \mathrm{E_t}\left[\left[\left(\widehat{R}_{t+1}(1-h_t) + v_t\widetilde{R}_{t+1}\right)\ h_{t+1}(v_{t+1})\right]^{-\gamma}\right]\ \widehat{R}_{t+1}$$

where $v_{t+1} = v_t\widetilde{R}_{t+1}/(\widehat{R}_{t+1}(1-h_t) + v_t\widetilde{R}_{t+1})$. I will assume a $\mathcal{C}^1$ solution exists for all t.

I also assume, as in the text, the independence of $\widehat{R}$ and $\widetilde{Q} \equiv \widetilde{R}/\widehat{R}$. In that case, an implication of the first order condition is that the expectation

$$\mathrm{E_t}\left[\left[\frac{h_t}{\left((1-h_t) + v_t\widetilde{Q}_{t+1}\right)\ h_{t+1}(v_{t+1})}\right]^{+\gamma}\right] \tag{1}$$

must not be a function of v_t. I use this fact to prove the following successive properties:

(i) $h_t' \leq h_t/v_t \quad \forall\ v_t, t.$

(ii) $-(1-h_t)/v_t \leq h_t' \quad \forall\ v_t, t.$

(iii) $0 < h_t' \quad \forall\ v_t, t.$

For the first point, assume the property holds for h_{t+1} but fails to hold for h_t. Write the expectation, above as

$$\mathrm{E_t}\left[\left[\frac{\frac{h_t(v_t)}{v_t}}{\frac{h_{t+1}(v_{t+1})}{v_{t+1}}}\right]^{+\gamma}\ \widetilde{Q}_{t+1}^{-\gamma}\right]. \tag{2}$$

The hypothesis implies that the derivative with respect to v_t of the numerator in the inner brackets is positive and the derivative with respect to v_{t+1} of the denominator is negative. Also, the derivative $\frac{dv_{t+1}}{dv_t}$ is

$$\frac{\widetilde{Q}_{t+1}}{((1-h_t)+v_t\widetilde{Q}_{t+1})^2}\,(1-h_t+v_t h_t').$$

The hypothesis on h_t implies that $(1-h_t+v_t h_t')\geq 1$. So $\frac{dv_{t+1}}{dv_t}$ is positive. Together, these observations imply that an increase in v_t will raise the numerator and lower the denominator of square bracket term in equation (2) for all values of the random variable $\widetilde{Q}_{t+1}$. Hence the expectation cannot be constant. The contradiction, combined with the fact that the final optimal policy is $h_T=1$, which satisfies the induction hypothesis, proves $h_t'\leq h_t/v_t$ for all t.

Next, assume $-(1-h_{t+1})/v_{t+1}\leq h_{t+1}'$ but that the reverse holds for h_t. Differentiate the denominator of equation (1) to get

$$\frac{1}{((1-h_t)+v_t\widetilde{Q}_{t+1})}\Big((\widetilde{Q}_{t+1}-h_t')((1-h_t)+v_t\widetilde{Q}_{t+1})\,h_{t+1}+\widetilde{Q}_{t+1}\,(1-h_t+v_t h_t')h_{t+1}'\Big)$$

Using the result just shown, $h_{t+1}'\leq h_{t+1}/v_{t+1}$. And, by the induction hypothesis, $(1-h_t+v_t h_t')<0$. So the smallest the term in large parentheses can be is

$$\begin{aligned}&((1-h_t)+v_t\widetilde{Q}_{t+1})\,h_{t+1}v_t^{-1}\Big((\widetilde{Q}_{t+1}-h_t')v_t+(1-h_t+v_t h_t')\Big)\\ &=((1-h_t)+v_t\widetilde{Q}_{t+1})\,h_{t+1}v_t^{-1}\,\Big((1-h_t)+v_t\widetilde{Q}_{t+1}\Big)>0.\end{aligned}$$

These observations imply that an increase in v_t will lower the numerator and raise the denominator of the bracketed term in (1) for all values of $\widetilde{Q}_{t+1}$. Hence the expectation cannot be constant. The contradiction, combined with the fact that the final optimal policy satisfies the induction hypothesis, proves $-(1-h_t)/v_t\leq h_t'$ for all t.

The third step proceeds similarly: assume the inequality (iii) holds for $t+1$ but not t. By the previous point (ii), we now have $(1-h_t+v_t h_t')\geq 0$ even though $h_t'<0$. This means $\frac{dv_{t+1}}{dv_t}$ is always positive. So an increase in v_t must increase the denominator and decrease the numerator of (1), contradicting the constancy of the expectation. Given that $h_T=1$, the constancy of the expectation at $T-1$ immediately implies that h_{T-1} must be strictly increasing. Hence h_{T-1} satisfies the induction hypothesis (iii). So we conclude $h_t'>0$ for all $t<T$.

Now the limit of the discrete time maps: $h\equiv\lim_{t\to-\infty}\,h_t$ must also satisfy the condition that

$$\mathrm{E_t}\left[\left[\frac{h(v_t)}{\Big((1-h)+v_t\widetilde{Q}_{t+1}\Big)\;h(v_{t+1})}\right]^{+\gamma}\right]$$

is constant. The limit of increasing functions cannot be decreasing. However it can be flat. But if $h()$ is constant, then an increase in v_t would still raise $((1-h_t)+v_t\widetilde{Q}_{t+1})$ and change the expectation. So we must also have $h'>0$.

QED

Proposition 4.1

Proof. The elasticity $\mathcal{I}$ can be computed from direct differentiation of $P^{(1)}$ in terms of f or g using the value-neutral condition $dG_t/dX^{(1)} = -P^{(1)}$ and

$$\frac{dv_t}{dX^{(1)}} = \frac{v_t}{X^{(1)}}(1+v_t g(v_t)) = \frac{v_t}{X^{(1)}}(1+f(v_t))$$

which follows from the definition $v = D^{(1)}X^{(1)}/G$. *QED*

Proposition 5.1

Proof. Let us distinguish, at each intervention date, between the times immediately before and immediately after the exchange, writing these as e.g., τ_k- and τ_k+. (It is immaterial whether allocation and consumption decisions are made before or after.) Also write the value function of the representative agent as $J_t = J(D_t^{(1)}, X_t^{(1)}, G_t)$. Recall the superscript denotes per-share values, so that the total dividend income of the agent is $D_t = D_t^{(1)}\, X_t^{(1)}$.

Let J_t^o be the value function of the equivalent economy in which no further interventions will take place, that is, in which $X_s^{(1)} = X_t^{(1)}$ for all $s \geq t$, as in the original model. Similarly, let J_t^k be the value function under the assumption that the kth exchange does *not* take place. Then the assumption of the proposition that agents are indifferent to each exchange can be expressed as $J^k_{\tau_k-} = J_{\tau_k+}$.

Now consider the value function at any date t such that $\tau_{K-1} < t \leq \tau_K$. This must satisfy

$$J_t = \max_{\{C_{t+n}\}_{n=0}^{\infty}} \mathrm{E_t}\left[\sum_{n=0}^{\infty} \beta^n\ u(C_{t+n})\right].$$

The expectation can be written

$$\sum_{j=0}^{\infty} \mathrm{E_t}\left[\sum_{n=0}^{j-1} \beta^n\ u(C_{t+n}) + \sum_{n=j}^{\infty} \beta^n\ u(C_{t+n}) | \tau_K = t+j\right] \mathrm{P}\left(\tau_K = t+j\right)$$

and the inner, conditional expectation is

$$\mathrm{E_t}\left[\sum_{n=0}^{j-1}\beta^n\ u(C_{t+n})+\mathrm{E}_{t+j}\left(\sum_{n=j}^{\infty}\beta^n\ u(C_{t+n})|\tau_K=t+j\right)|\tau_K=t+j\right].$$

Define the conditionally optimal policy $\left\{C^j_{t+n}\right\}_{n=1}^{\infty}$ to be the one that maximizes this latter expectation. But, by assumption, the solution to

$$\max_{\{C_{t+n}\}_{n=j}^{\infty}}\mathrm{E}_{t+j}\left(\sum_{n=j}^{\infty}\beta^n\ u(C_{t+n})|\tau_K=t+j\right)$$

coincides with the same value in the absence of the exchange, which is

$$\max_{\{C_{t+n}\}_{n=j}^{\infty}}\mathrm{E}_{t+j}\left(\sum_{n=j}^{\infty}\beta^n\ u(C_{t+n})|\text{no trade at any date } s\geq t+j\right).$$

This solution is what was called J^K_{t+j} (times β^{j-1}).

Hence the conditionally optimal policy solves

$$\max_{\{C_{t+n}\}_{n=0}^{\infty}}\mathrm{E_t}\left[\sum_{n=0}^{j-1}\beta^n\ u(C_{t+n})|\text{no trade at any date } s<t+j\right]+\beta^{j-1}\ J^K_{t+j}$$

which is the same as J^K_t. Since this is true regardless of the conditioning index j, it follows that $J_t = J^K_t$ and also that the optimizing policies of the two problems coincide. Since K is the last exchange date, we also have that $J_t = J^o_t$ by definition of the latter.

We have shown that the value function and optimal policies are the same at $\tau_{K-1} < t \leq \tau_K$ as they would be if there were no intervention. By backward induction, the same argument applies at $\tau_{K-2} < t \leq \tau_{K-1}$, and so on. Since the optimal policies and value function are the same functions as in the no-intervention economy, it follows that the pricing function, and its derivatives must also be the same.

QED

B. $\mathcal{I}$ as a Function of v: The Two-period Case.

This appendix analyzes the dependence of $\mathcal{I}$ on v in a two-period CRRA economy analogous to the two-period pure-endowment example in Section 4. This illustrates how the availability of savings dampens both the current consumption effect and the future consumption effect.

Suppose the representative agent has G_0 goods today and will receive a dividend XD_1 next period which he consumes and then dies. (I suppress the per-share superscript here to lighten the notation.) If the economy does not have a savings technology, so that $C_0 = G_0$, then, as in the example in the text, we have

$$\mathcal{I} = \gamma \left[1 + \frac{XP}{C_0}\right] = \gamma \left[1 + \frac{XP}{G_0}\right].$$

Now add the possibility of risklessly storing goods (with gross return R) and, for simplicity, fix the agent's savings to be $(1-\lambda)G_0$ or $C_0 = \lambda G_0$ for some fraction λ. Ignore the determination of the optimal λ and view the price per share today as $P(\lambda)$. The algebra in calculating $\mathcal{I}$ is a little messy, but worthwhile.

First,

$$P = P(G_0, X; \lambda) = \lambda^{\gamma}\ G_0^{\gamma}\ \beta \mathrm{E}[(XD_0\widetilde{R}_1 + (1-\lambda)RG_0)^{-\gamma}\ D_0\widetilde{R}_1]$$

where I have written $D_1 = D_0\widetilde{R}_1$, as before. Differentiating this with respect to X subject to $dG_0/dX = -P$ and scaling by X/P produces three terms in $\mathcal{I}$.

$$\begin{aligned}
\text{Term I}: \quad & \gamma\ D_0X\lambda^{\gamma}\ G_0^{\gamma-1}\ \beta\mathrm{E}(XD_0\widetilde{R}_1 + (1-\lambda)RG_0)^{-\gamma}\widetilde{R}_1 \\
= \quad & \gamma\ v_0\lambda^{\gamma}\ \ \beta\mathrm{E}(v_0\widetilde{R}_1 + (1-\lambda)R)^{-\gamma}\widetilde{R}_1 \\[1ex]
\text{Term II}: \quad & -\gamma\ D_0X\lambda^{\gamma}\ G_0^{\gamma}(1-\lambda)R\ \beta\mathrm{E}(XD_0\widetilde{R}_1 + (1-\lambda)RG_0)^{-\gamma-1}\widetilde{R}_1 \\
= \quad & -\gamma\ v_0\lambda^{\gamma}\ (1-\lambda)R\ \beta\mathrm{E}(v_0\widetilde{R}_1 + (1-\lambda)R)^{-\gamma-1}\widetilde{R}_1 \\[1ex]
\text{Term III}: \quad & \gamma\ D_0^2X\lambda^{\gamma}\ G_0^{\gamma}\ \beta\mathrm{E}(XD_0\widetilde{R}_1 + (1-\lambda)RG_0)^{-\gamma-1}\widetilde{R}_1^2/P \\
= \quad & \gamma\ v_0\ \frac{\mathrm{E}(v_0\widetilde{R}_1 + (1-\lambda)R)^{-\gamma-1}\widetilde{R}_1^2}{\mathrm{E}(v_0\widetilde{R}_1 + (1-\lambda)R)^{-\gamma}\widetilde{R}_1}. \qquad (3)
\end{aligned}$$

The first term can also be written as simply $\gamma XP(\lambda)/G_0$, which shows that it corresponds to the current consumption effect. But now it is also clear, from the second line, that this term declines as λ does. As the agent saves more, the impact of the value-neutral trade on current consumption is, of course, smaller.

The second term is another contribution from the effect of altering current consumption, which arises because now (unlike in the endowment model) an increase in consumption goods today produces interest income next period. Comparing the second line with the fourth, term II is strictly smaller than term I, because the only difference is an extra factor in the expectation of term II equal to $\Psi \equiv (1-\lambda)R/(v_0\widetilde{R}_1+(1-\lambda)R) \leq 1$. Calling the term I integrand Γ, the two terms can also be combined, giving

$$\gamma\frac{XP}{G_0}\left[1-\frac{\mathrm{E}\Psi\ \Gamma}{\mathrm{E}\Gamma}\right]$$

$$= \gamma\ \lambda^{\gamma}\ \beta\mathrm{E}\Gamma\Theta = \gamma\ v_0^2\lambda^{\gamma}\ \beta\mathrm{E}(v_0\widetilde{R}_1+(1-\lambda)R)^{-\gamma-1}\widetilde{R}_1^2$$

where $\Theta \equiv (1-\Psi) = v_0\widetilde{R}_1/(v_0\widetilde{R}_1+(1-\lambda)R)$. And the last equation clearly still decreases as λ does, vanishing at $\lambda = 0$. As one would expect, the total impact of the current consumption terms is smaller when the current consumption ratio is lower.

Finally, term III is what I referred to as the future consumption effect in the earlier examples. It can be reexpressed as

$$\gamma\ \frac{\mathrm{E}\Theta\ \Gamma}{\mathrm{E}\Gamma} \leq \gamma.$$

Again, the ability to store goods lowers this term relative to the endowment economy cases. Intuitively, the presence of positive savings at date zero lowers the percentage impact of a change in dividends on future marginal utility. Somewhat less obviously, term III also decreases as λ does, regardless of the parameter values.[17] Loosely, this is due to the extra term in the numerator, Θ, which behaves like $(1-\lambda)^{-1}$.

Hence all the terms in $\mathcal{I}$ are less than their counterparts in the corresponding endowment economy, and more so as λ declines. Now recall that the *optimal* consumption ratio is $\lambda = h(v)$, which is an increasing function of v. This reveals the mechanism that causes the liquidity of the market for asset one to increase as the level of liquid balances in the economy does.[18] The reason this happens is because the propensity to consume current wealth increases as liquid wealth declines, and this propensity, in turn, determines how big an impact a change in risky asset holdings has on marginal utilities. That impact dictates the willingness of agents to accommodate trades, or the rigidity of prices.

[17]This can be shown using the result that two monotonically related random variables must be positively correlated. The proof is available upon request.

[18]I have not proven, even in the two-period case, that $\mathcal{I}(v)$ must be increasing in v. The argument above does not consider either the variation of λ with v or the direct effect, i.e. not through the savings term, of v on $\mathcal{I}$. What the argument shows is that, whichever direction these other terms go, the savings channel always makes illiquidity rise with v.

References

Caballero, R. J., 1990, "Consumption Puzzles and Precautionary Savings," *Journal of Monetary Economics*, 25, 113–136.

Carroll, C. D., 1992, "The Buffer-Stock Theory of Saving: Some Macroeconomic Evidence," *Brookings Papers on Economic Activity*, pp. 61–156.

Carroll, C. D., 2004, "Theoretical Foundations of Buffer Stock Savings," Johns Hopkins University Working Paper.

Carroll, C. D., and M. S. Kimball, 1996, "On the Concavity of the Consumption Function," *Econometrica*, 64, 981–992.

Chordia, T., A. Sarkar, and A. Subrahmanyam, 2005, "An Empirical Analysis of Stock and Bond Market Liquidity," *Review of Financial Studies*, 18, 85–129.

Clarida, R. H., 1987, "Consumption, Liquidity Constraints, and Asset Accumulation in the Face of Random Fluctuations in Income," *International Economic Review*, 28, 339–351.

Deaton, A. S., 1991, "Saving and Liquidity Constraints," *Econometrica*, 59, 1221–1248.

Fujimoto, A., 2004, "Macroeconmic Sources of Systematic Liquidity," University of Alberta Working Paper.

Johnson, T. C., 2006, "Dynamic Liquidity in Endowment Economies," *Journal of Financial Economics*, 80, 531–562.

Johnson, T. C., 2007, "Volume, Liquidity, and Liquidity Risk," *Journal of Financial Economics*, fothcoming.

Lucas, R., 1978, "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1429–1445.

Pagano, M., 1989, "Trading Volume and Asset Liquidity," *Quarterly Journal of Economics*, 104, 255–274.

Szeidl, A., 2002, "On Ergodic Distributions and Buffer Stock Savings Models," Harvard University Working Paper.